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On the temporal order of first-passage times in one-dimensional lattice random walks

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Abstract

A random walk problem with particles on discrete double infinite linear grids is discussed. The model is based on the work of Montroll and others. A probability connected with the problem is given in the form of integrals containing modified Bessel functions of the first kind. By using several transformations, simpler integrals are obtained from which for two and three particles asymptotic approximations are derived for large values of the parameters. Expressions of the probability for n particles are also derived.

I returned and saw under the sun, that the race is not to the swift, nor the battle to the strong, neither yet bread to the wise, nor yet riches to men of understanding, nor yet favour to men of skill; but time and chance happeneth to them all. George Orwell, *Politics and the English Language*, Selected Essays, Penguin Books, 1957. (The citation is from Ecclesiastes 9:11.)

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1. Introduction

The subject of random motion is one on which an enormous amount of mathematical studies have been made. We mention in this respect the classical work of Rayleigh, Smoluchowski, Chandrasekhar, and countless others [6]. In this paper, we are interested in the specialization of this general notion to random walk on a periodic lattice, where a particle makes random jumps between neighbouring sites of this lattice. In this respect we refer in particular to the pioneering work by Montroll and his collaborators which has provided the inspiration for the present work.

We shall very briefly indicate the method of Montroll’s approach, where throughout this paper we shall limit ourselves to random walks on one (or more) linear (1D) lattice chains. We shall also suppose that the jump probabilities of a random walker to the left and to the right are equal, and hence equal to $p = \frac{1}{2}$. Initially, the time is considered to be discrete, which means that we consider the situation of the particle after a discrete number of jumps n , which is equivalent to allowing the particle to jump once in every unit of time. Montroll et al. [4,5] now introduce two quantities which are of very great importance. These are

1. $P_n(\ell)$, the probability that the random walker will be at site ℓ after the n th jump.
2. $f_n(\ell)$, the probability that the random walker will be at site ℓ after the n th jump *for the first time*.

Of course, it is assumed that before the first jump ($n = 0$) the particle is at the origin ($\ell = 0$).

The function $P_n(\ell)$ satisfies the following equation:

$$P_n(\ell) = \frac{1}{2}P_{n-1}(\ell - 1) + \frac{1}{2}P_{n-1}(\ell + 1). \tag{1.1}$$

(If at epoch $n - 1$, the particle is at either $\ell - 1$ or $\ell + 1$, it will have a probability $p = \frac{1}{2}$ to be at ℓ at epoch n . If it is anywhere else at epoch $n - 1$, its chance of being at ℓ one jump later, is zero.) This equation also shows that the random walk, as described above, is a Markoff process, in that the state (ℓ) of the random walker at a given epoch n depends only on that at *one* moment earlier.

Montroll then introduces a generating function $U(\ell, z) = \sum_{n=0}^{\infty} P_n(\ell)z^n$. This function $U(\ell, z)$ is then calculated explicitly, from which P_n and various moments over ℓ can be calculated. For details we refer to [4] and [5]. We also refer to these papers for the treatment of the first passage times $f_n(\ell)$ and the corresponding generating function $F(\ell, z) = \sum_{n=0}^{\infty} f_n(\ell)z^n$. The quantity $f_n(\ell)$ is the probability of reaching the site ℓ for the first time at the n th jump.

For the sake of completeness we give the explicit expressions for $U(\ell, z)$ and $F(\ell, z)$:

$$\begin{aligned}
 U(\ell, z) &= \left(\frac{z}{2}\right)^\ell {}_2F_1\left(\frac{\ell + 1}{2}, \frac{\ell + 2}{2}; \ell + 1; z^2\right) = \frac{1}{\sqrt{1 - z^2}} \left(\frac{1 - \sqrt{1 - z^2}}{z}\right)^\ell, \\
 F(\ell, z) &= \left(\frac{z}{2}\right)^\ell {}_2F_1\left(\frac{\ell}{2}, \frac{\ell + 1}{2}; \ell + 1; z^2\right) - \delta_{\ell,0}\sqrt{1 - z^2} = \frac{U(\ell, z) - \delta_{\ell,0}}{U(0, z)},
 \end{aligned}
 \tag{1.2}$$

from which explicit forms of $P_n(\ell)$ and $f_n(\ell)$ follow.

Montroll et al. [5] also present a method of treating the time as a continuous variable. Then we introduce as fundamental quantities the following probability densities:

$$\begin{aligned} \overline{P}(\ell, t) dt &: \text{the probability density for the random walker to be at } \ell \text{ during interval } (t, t + dt). \\ \overline{F}(\ell, t) dt &: \text{the probability density for the random walker to arrive at } \ell \text{ during interval } (t, t + dt) \\ &\text{for the first time.} \end{aligned} \tag{1.3}$$

Jumps are now taken to occur at random times t_1, t_2, t_3, \dots . This implies the introduction of the random variables $T_1 = t_1, T_2 = t_2 - t_1, \dots, T_n = t_n - t_{n-1}$, which have the common density $\psi(t)$. For $\psi(t)$ we take the exponential density $\psi(t) = \alpha e^{-\alpha t}$, where α is the average number of jumps made by the random walker per unit of time. From this point on we shall concentrate on the first-passage probability density function, that being the one which we shall need most in the following applications.

We also introduce the probability densities

$$\psi_0(t) = \delta(t), \quad \psi_n(t) = \int_0^t \psi(t - \tau)\psi_{n-1}(\tau) d\tau, \quad n = 1, 2, 3, \dots \tag{1.4}$$

The function $\psi_n(t)$ can be interpreted as the probability density that the n th jump of the random walker takes place in the time interval $(t, t + dt)$. We have

$$\psi_n(t) = \alpha^n e^{-\alpha t} \frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \dots \tag{1.5}$$

It can now easily be understood that [5]

$$\overline{F}(\ell, t) = \sum_{n=0}^{\infty} f_n(\ell)\psi_n(t). \tag{1.6}$$

If we use the $\psi_n(t)$ given above and the $f_n(\ell)$ that follow from the second line of (1.2), we obtain

$$\overline{F}(\ell, t) = 2^{-\ell} e^{-\alpha t} t^{-1} \sum_{n=0}^{\infty} \frac{(\ell/2)_n (\ell/2 + 1/2)_n}{n! (\ell + 1)_n} \frac{(\alpha t)^{\ell+2n}}{(\ell + 2n - 1)!}, \tag{1.7}$$

where $(a)_n$ denotes Pochhammer’s symbol defined by

$$(a)_0 = 1, \quad (a)_n = a(a + 1) \cdots (a + n - 1), \quad n = 1, 2, 3, \dots \tag{1.8}$$

Comparing the expansion in (1.7) with that of the modified Bessel function of the first kind, see [1, Ch. 9],

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n! \Gamma(\nu + n + 1)}, \tag{1.9}$$

and using the duplication formula of the gamma function

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \tag{1.10}$$

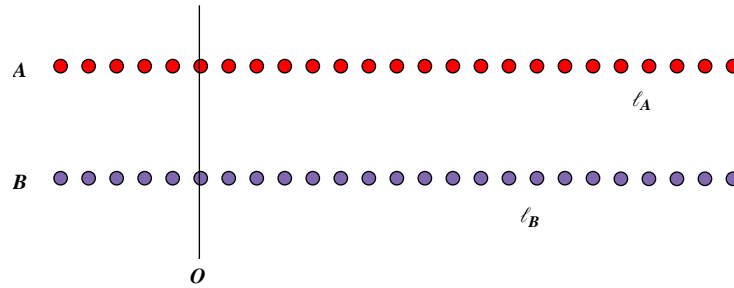


Fig. 1. The random walks for A and B.

an explicit form for $\bar{F}(\ell, t)$ is obtained:

$$\bar{F}(\ell, t) = e^{-\alpha t} \frac{\ell}{t} I_\ell(\alpha t). \tag{1.11}$$

From [1, Eq. 11.4.13] it follows that for $\ell \neq 0$

$$\int_0^\infty \bar{F}(\ell, t) dt = \ell \int_0^\infty e^{-\alpha t} I_\ell(\alpha t) \frac{dt}{t} = 1. \tag{1.12}$$

It is of interest to consider the problem of several simultaneous random walkers on a lattice chain, and the behaviour in time of their mutual configuration. It is as an introduction to this work that we shall consider two, three, . . . , independent random walkers on separate lattice chains. We begin with two random walkers and consider the situation as shown in Fig. 1.

Remark 1. The integers ℓ_A and ℓ_B may separately assume negative values. However, to avoid the use of absolute value signs, we consider only positive values of ℓ_A and ℓ_B . But all results hold for negative values when we replace these quantities by their absolute values.

We are interested in finding the probability that particle A arrives at ℓ_A before particle B arrives at ℓ_B . The solution to this problem is an intermediate result for the treatment of a 1D-grid random walk problem of an agglomeration of many particles.

We know that the probability density for A to arrive for the first time at ℓ_A in the interval $(t, t + dt)$ is

$$\bar{F}(\ell_A, t) dt = \frac{\ell_A}{t} e^{-\alpha t} I_{\ell_A}(\alpha t) dt. \tag{1.13}$$

It is now obvious that the desired probability can be written as

$$P(t_{\ell_A} \leq t_{\ell_B}) = \int_0^\infty dt_A \bar{F}(\ell_A, t_A) \int_{t_A}^\infty dt_B \bar{F}(\ell_B, t_B), \tag{1.14}$$

where t_{ℓ_A} is the time that particle A reaches the site ℓ_A for the first time, and similar for t_{ℓ_B} . The independence of the walkers is expressed by the fact that it is the product of two \bar{F} -functions which is being integrated.

Using (1.12) we have

$$P(t_{\ell_A} \leq t_{\ell_B}) = 1 - \int_0^\infty dt_A \bar{F}(\ell_A, t_A) \int_0^{t_A} dt_B \bar{F}(\ell_B, t_B) \tag{1.15}$$

and interchanging the order of integration in this integral, we derive the symmetry properties (which are evident from the random walk problem)

$$P(t_{\ell_A} \leq t_{\ell_B}) = 1 - P(t_{\ell_B} \leq t_{\ell_A}), \quad \text{hence} \quad P(t_{\ell_A} = t_{\ell_B}) = \frac{1}{2}, \quad \text{if } \ell_A = \ell_B. \tag{1.16}$$

If $\ell_A = \ell_B$ we can also use integration by parts

$$\begin{aligned} \int_0^\infty dt_A \bar{F}(\ell, t_A) \int_{t_A}^\infty dt_B \bar{F}(\ell, t_B) &= - \int_0^\infty \left[\int_{t_A}^\infty \bar{F}(\ell, \tau) d\tau \right] d \left[\int_{t_A}^\infty \bar{F}(\ell, \tau) d\tau \right] \\ &= \frac{1}{2} \left[\int_0^\infty \bar{F}(\ell, \tau) d\tau \right]^2 = \frac{1}{2}. \end{aligned} \tag{1.17}$$

In this paper we derive asymptotic expansions of $P(t_{\ell_A} \leq t_{\ell_B})$ given by

$$P = \ell_A \ell_B \int_0^\infty \frac{dt_A}{t_A} e^{-t_A} I_{\ell_A}(t_A) \int_{t_A}^\infty \frac{dt_B}{t_B} e^{-t_B} I_{\ell_B}(t_B). \tag{1.18}$$

In (1.18), the scale factor α has been absorbed in t_A and t_B , because of $\frac{dt}{t} = \frac{d\alpha t}{\alpha t}$.

We will give one expansion that holds for large values of ℓ_A and one for the case that both parameters ℓ_A and ℓ_B are large. We also give an expansion that holds just when the sum $\ell_A + \ell_B$ is large.

2. Transforming the integral

We study the integral (1.18). We use well-known properties of the modified Bessel function to transform the double integral in (1.18) into a single integral.

The inner integral in (1.18) can be modified by evaluating

$$S_\ell(t) := \ell \int_t^\infty e^{-s} I_\ell(s) \frac{ds}{s}. \tag{2.1}$$

where $\ell = 1, 2, \dots$. We use the integral representation (see [1, Eq. 9.6.19])

$$I_n(s) = \frac{1}{\pi} \int_0^\pi e^{s \cos \theta} \cos n\theta d\theta \tag{2.2}$$

for integer values of n . Integrating by parts we obtain

$$\frac{\ell}{s} I_\ell(s) = \frac{1}{\pi} \int_0^\pi e^{s \cos \theta} \sin \theta \sin \ell\theta d\theta. \tag{2.3}$$

It follows that

$$\begin{aligned} \ell \int_t^\infty e^{-ps} I_\ell(s) \frac{ds}{s} &= \frac{1}{\pi} \int_0^\pi d\theta \sin \theta \sin \ell\theta \int_t^\infty ds e^{-s(p-\cos \theta)} \\ &= \frac{e^{-pt}}{\pi} \int_0^\pi \frac{\sin \theta \sin \ell\theta}{p - \cos \theta} e^{t \cos \theta} d\theta, \end{aligned} \tag{2.4}$$

which holds for $p \geq 1$. It follows that $S_\ell(t)$ of (2.1) can be written as

$$S_\ell(t) = \frac{e^{-t}}{\pi} \int_0^\pi \frac{\sin \theta \sin \ell\theta}{1 - \cos \theta} e^{t \cos \theta} d\theta = \frac{e^{-t}}{\pi} \int_0^\pi \cot \frac{1}{2}\theta \sin \ell\theta e^{t \cos \theta} d\theta. \tag{2.5}$$

Using this relation and interchanging the order of integration in (1.18), we obtain

$$\begin{aligned} P &= \ell_A \int_0^\infty e^{-t} I_{\ell_A}(t) S_{\ell_B}(t) \frac{dt}{t} \\ &= \frac{\ell_A}{\pi} \int_0^\pi \cot \frac{1}{2}\theta \sin \ell_B \theta \left[\int_0^\infty e^{-2t+t \cos \theta} I_{\ell_A}(t) \frac{dt}{t} \right] d\theta. \end{aligned} \tag{2.6}$$

Invoking again (2.3) we obtain

$$P = \frac{1}{\pi^2} \int_0^\pi d\theta_2 \frac{\sin \theta_2 \sin \ell_B \theta_2}{1 - \cos \theta_2} \int_0^\pi d\theta_1 \frac{\sin \theta_1 \sin \ell_A \theta_1}{(1 - \cos \theta_1) + (1 - \cos \theta_2)}. \tag{2.7}$$

The θ_1 -integral can be evaluated. An easier way is to use in (2.6) the Laplace integral

$$\ell \int_0^\infty e^{-pt} I_\ell(t) \frac{dt}{t} = \left(p + \sqrt{p^2 - 1} \right)^{-\ell}, \quad \ell > 0, \quad p \geq 1, \tag{2.8}$$

which follows from [1, 29.3.53] by taking $a = 1, b = -1$. This gives

$$P = \frac{1}{\pi} \int_0^\pi \cot \frac{1}{2}\theta \sin \ell_B \theta \left(p + \sqrt{p^2 - 1} \right)^{-\ell_A} d\theta, \quad p = 2 - \cos \theta. \tag{2.9}$$

3. Asymptotic expansions

We give three asymptotic expansions:

- one for large ℓ_A , with ℓ_B fixed, or small,
- one for large ℓ_A and ℓ_B , with $\ell_A \sim \ell_B$,
- one uniform expansion in which one or both parameters may be large.

3.1. The case $\ell_A \gg \ell_B$

We start from (2.9) in the form

$$P = \frac{1}{\pi} \int_0^\pi f(\theta) e^{-\ell_A \phi(\theta)} d\theta, \tag{3.1}$$

where

$$f(\theta) = \cot \frac{1}{2}\theta \sin \ell_B \theta, \quad \phi(\theta) = \ln \left(p + \sqrt{p^2 - 1} \right), \quad p = 2 - \cos \theta. \tag{3.2}$$

First we observe that

$$\phi'(\theta) = \frac{\sin \theta}{\sqrt{p^2 - 1}} = \frac{\cos \frac{1}{2}\theta}{\sqrt{1 + \sin^2 \frac{1}{2}\theta}}. \tag{3.3}$$

Hence, $\phi(\theta)$ is an increasing function on $[0, \pi]$ with

$$\phi(0) = 0, \quad \phi'(0) = 1, \quad \phi'(\pi) = 0. \tag{3.4}$$

It follows that

$$P \sim \frac{1}{\pi} \int_0^{\theta_0} f(\theta) e^{-\ell_A \phi(\theta)} d\theta, \tag{3.5}$$

where θ_0 is a fixed number in $(0, \pi)$, and the error in this approximation is exponentially small when ℓ_A is large.

Carrying out an integration by parts in the form

$$P \sim \frac{-1}{\pi \ell_A} \int_0^{\theta_0} \frac{f(\theta)}{\phi'(\theta)} d e^{-\ell_A \phi(\theta)} \tag{3.6}$$

leads to

$$P \sim \frac{-1}{\pi \ell_A} \frac{f(\theta)}{\phi'(\theta)} e^{-\ell_A \phi(\theta)} \Big|_{\theta=0}^{\theta=\theta_0} + \frac{1}{\pi \ell_A} \int_0^{\theta_0} f_1(\theta) e^{-\ell_A \phi(\theta)} d\theta, \tag{3.7}$$

where

$$f_1(\theta) = \frac{d}{d\theta} \frac{f(\theta)}{\phi'(\theta)}. \tag{3.8}$$

We can repeat this procedure, and compute the integrated terms. The terms at θ_0 can be neglected because they give exponentially small contributions compared with the contributions from $\theta = 0$. Note that we cannot take $\theta_0 = \pi$, because $\phi'(\pi) = 0$.

In this way we obtain the asymptotic expansion

$$P \sim \frac{1}{\pi \ell_A} \left[a_0 + \frac{a_1}{\ell_A} + \frac{a_2}{\ell_A^2} + \dots \right], \tag{3.9}$$

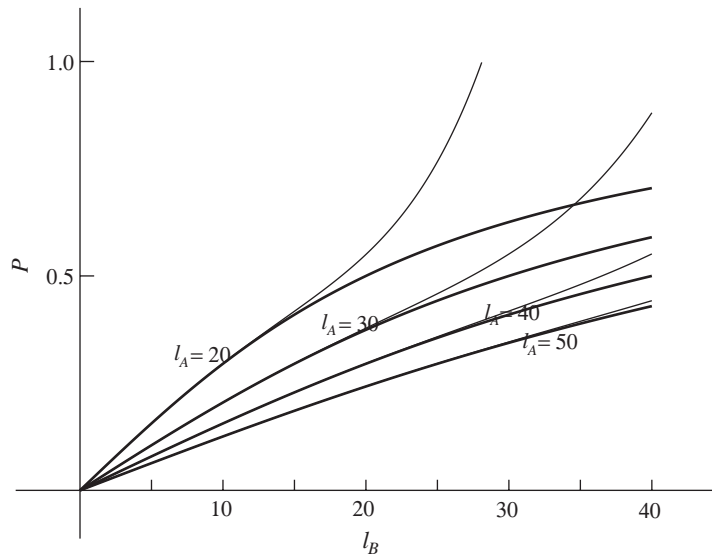


Fig. 2. Graphs of $P(t_{\ell_A} \leq t_{\ell_B})$ based on the asymptotic approximation (3.9) (thin curves), compared with graphs based on expansion (3.34) (thick curves). The thin and thick graphs deviate from each other because of the failure of the non-uniform approximations for large values of ℓ_B .

where, for $k = 0, 1, 2, \dots$,

$$a_k = \frac{f_k(0)}{\phi'(0)}, \quad f_{k+1}(\theta) = \frac{d}{d\theta} \frac{f_k(\theta)}{\phi'(\theta)}, \quad f_0(\theta) = f(\theta). \tag{3.10}$$

The coefficients a_k with odd indices are zero. This follows from observing that $f(\theta)$ and $\phi'(\theta)$ are even functions; see (3.2) and (3.3). Hence, $f_1(\theta)$ of (3.8) is odd. By using the recursion in (3.10) it follows that $f_{2k}(\theta)$ is even, and that $f_{2k+1}(\theta)$ is odd. The first non-zero coefficients are

$$a_0 = 2\ell_B, \quad a_2 = \frac{2}{3}\ell_B(1 - \ell_B^2), \quad a_4 = \frac{1}{30}\ell_B(23 - 80\ell_B^2 + 12\ell_B^4). \tag{3.11}$$

In Fig. 2 we compare the approximations based on (3.9) with values obtained by using the expansion in (3.34), which holds when $\ell_A + \ell_B$ is large. We see that for smaller values of ℓ_B the graphs of the asymptotic approximation (3.9) are in agreement with the graphs obtained from the expansion that holds when at least one of the parameters ℓ_A or ℓ_B is large. The failure of the non-uniform approximations (shown as thin curves) is due to the failure of the asymptotic approximation (3.9) that has been chosen for this case.

3.2. The case $\ell_A \sim \ell_B$, both large

We replace in (2.9) ℓ_A by ℓ and ℓ_B by $\ell + \delta$. We know that $P = \frac{1}{2}$ if $\delta = 0$. We expand (2.9) for large values of ℓ , keeping δ fixed. We have

$$P = \frac{1}{2} + P_1 + P_2, \tag{3.12}$$

where

$$\begin{aligned}
 P_1 &= \frac{1}{\pi} \int_0^\pi \cot \frac{1}{2}\theta (\cos \delta\theta - 1) \sin \ell\theta \left(p + \sqrt{p^2 - 1}\right)^{-\ell} d\theta \\
 &= \Im \frac{1}{\pi} \int_0^\pi \cot \frac{1}{2}\theta (\cos \delta\theta - 1) e^{i\ell\theta} \left(p + \sqrt{p^2 - 1}\right)^{-\ell} d\theta, \\
 P_2 &= \frac{1}{\pi} \int_0^\pi \cot \frac{1}{2}\theta \sin \delta\theta \cos \ell\theta \left(p + \sqrt{p^2 - 1}\right)^{-\ell} d\theta \\
 &= \Re \frac{1}{\pi} \int_0^\pi \cot \frac{1}{2}\theta \sin \delta\theta e^{i\ell\theta} \left(p + \sqrt{p^2 - 1}\right)^{-\ell} d\theta.
 \end{aligned} \tag{3.13}$$

Let

$$g(\theta) = \cot \frac{1}{2}\theta (\cos \delta\theta - 1), \quad \phi(\theta) = -i\theta + \ln \left(p + \sqrt{p^2 - 1}\right). \tag{3.14}$$

Then we integrate by parts in the integral for P_1

$$\frac{1}{\pi} \int_0^\pi g(\theta) e^{-\ell\phi(\theta)} d\theta = -\frac{1}{\pi\ell} \int_0^\pi \frac{g(\theta)}{\phi'(\theta)} d e^{-\ell\phi(\theta)}, \tag{3.15}$$

and we obtain an expansion as in (3.9),

$$P_1 \sim \frac{1}{\pi\ell} \left[b_0 + \frac{b_1}{\ell} + \frac{b_2}{\ell^2} + \dots \right], \tag{3.16}$$

where, for $k = 0, 1, 2, \dots$,

$$b_k = \Im \frac{g_k(0)}{\phi'(0)}, \quad g_{k+1}(\theta) = \frac{d}{d\theta} \frac{g_k(\theta)}{\phi'(\theta)}, \quad g_0(\theta) = g(\theta). \tag{3.17}$$

It turns out that the coefficients with even indices are zero. To verify this we can use a similar argument as for the a_k in (3.10). The first non-zero coefficients are

$$b_1 = -\frac{1}{2}\delta^2, \quad b_3 = \frac{1}{4}\delta^2, \quad b_5 = \frac{1}{96}\delta^2(4\delta^4 - 20\delta^2 - 77). \tag{3.18}$$

In a similar way, let $h(\theta) = \cot \frac{1}{2}\theta \sin \delta\theta$. Then

$$P_2 \sim \frac{1}{\pi\ell} \left[c_0 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \dots \right], \tag{3.19}$$

where, for $k = 0, 1, 2, \dots$,

$$c_k = \Re \frac{h_k(0)}{\phi'(0)}, \quad h_{k+1}(\theta) = \frac{d}{d\theta} \frac{h_k(\theta)}{\phi'(\theta)}, \quad h_0(\theta) = h(\theta). \tag{3.20}$$

It turns out that the coefficients with even indices are zero. The first non-zero coefficients are

$$c_0 = \delta, \quad c_2 = \frac{1}{6}\delta(\delta^2 - 1), \quad c_4 = -\frac{1}{240}\delta^2(12\delta^4 + 20\delta^2 - 77). \tag{3.21}$$

3.3. The case $\ell_A + \ell_B$ large

Because of (see (3.4))

$$e^{-\ell_A \phi(\theta)} \sim e^{-\ell_A \theta}, \quad \theta \rightarrow 0, \tag{3.22}$$

we have for large values of ℓ_A

$$P \sim \frac{2}{\pi} \int_0^\infty \frac{\sin \ell_B \theta}{\theta} e^{-\ell_A \theta} d\theta = \frac{2}{\pi} \arctan \frac{\ell_B}{\ell_A}, \tag{3.23}$$

where we used [1, Eq. 29.3.110].

Observe that this estimate perfectly reflects the properties of P mentioned in (1.16); also, it is less than unity, as the probability P itself is. Moreover, in this estimate large values of ℓ_B do not disturb the approximation.

Result (3.23) is obtained by combining the dominant behaviour of $e^{-\ell_A \phi(\theta)}$ near the origin with the complete form $\sin \ell_B \theta$, without expanding this function.

We modify the integration by parts procedure of Section 3.1, by including the (possible large) parameter ℓ_B in the “phase function” $\phi(\theta)$. We can do this by writing $\sin \ell_B \theta = \Im e^{i\ell_B \theta}$. A complication is the pole of the function $\cot \frac{1}{2}\theta$, which singularity is removable in combination with the function $\sin \ell_B \theta$.

To perform the integration by parts procedure we proceed in the following way. In (2.9) we can consider ℓ_B as a continuous parameter, and we can differentiate with respect to ℓ_B . We also observe that P vanishes with ℓ_B . We have

$$\frac{\partial P}{\partial \ell_B} = \frac{1}{\pi} \Re \left[\int_0^\pi \theta \cot \frac{1}{2}\theta e^{i\ell_B \theta} \left(p + \sqrt{p^2 - 1} \right)^{-\ell_A} d\theta \right]. \tag{3.24}$$

We write this in the form

$$\frac{\partial P}{\partial \ell_B} = \frac{2}{\pi} \Re Q, \tag{3.25}$$

where

$$Q = \int_0^\pi f(\theta) e^{\psi(\theta)} d\theta, \tag{3.26}$$

with

$$f(\theta) = \frac{1}{2}\theta \cot \frac{1}{2}\theta, \quad \psi(\theta) = i\ell_B \theta - \ell_A \ln \left(p + \sqrt{p^2 - 1} \right). \tag{3.27}$$

We integrate by parts, starting with

$$Q = \int_0^\pi \frac{f(\theta)}{\psi'(\theta)} d e^{\psi(\theta)} = \frac{f(\theta)}{\psi'(\theta)} e^{\psi(\theta)} \Big|_{\theta=0}^{\theta=\pi} + \int_0^\pi f_1(\theta) e^{\psi(\theta)} d\theta, \tag{3.28}$$

where

$$f_1(\theta) = -\frac{d}{d\theta} \frac{f(\theta)}{\psi'(\theta)}. \tag{3.29}$$

We repeat this procedure, and compute the integrated terms. Again, the terms at $\theta = \pi$ can be neglected.

We obtain

$$Q \sim d_0 + d_1 + d_2 + \dots, \tag{3.30}$$

where

$$d_k = -\frac{f_k(0)}{\psi'(0)}, \quad f_k(\theta) = -\frac{d}{d\theta} \frac{f_{k-1}(\theta)}{\psi'(\theta)}, \quad k = 0, 1, 2, \dots, \tag{3.31}$$

and $f_0(\theta) = f(\theta)$. Again, all coefficients with odd index vanish. This follows from

$$\psi'(\theta) = i\ell_B - \ell_A \frac{\sin \theta}{\sqrt{p^2 - 1}} = i\ell_B - \ell_A \frac{\cos 1/2\theta}{\sqrt{1 + \sin^2 1/2\theta}}, \tag{3.32}$$

which is an even function and $f(\theta)$ is also even. Hence, $f_1(\theta)$ in (3.29) is odd; and so on.

We have

$$d_0 = \frac{1}{\ell_A - i\ell_B}, \quad d_2 = \frac{2\ell_A + i\ell_B}{6(\ell_A - i\ell_B)^4}, \quad d_4 = \frac{23\ell_A^2 + 129i\ell_A\ell_B + 2\ell_B^2}{60(\ell_A - i\ell_B)^7}. \tag{3.33}$$

Considering (3.25), taking the real parts of the coefficients and integrating the real parts over the interval $[0, \ell_B]$, we find

$$P \sim \frac{2}{\pi}(e_0 + e_2 + e_4 + e_6 \dots), \tag{3.34}$$

where

$$e_{2k} = \int_0^{\ell_B} d_{2k}(\ell'_B) d\ell'_B. \tag{3.35}$$

The first few are

$$\begin{aligned} e_0 &= \arctan \frac{\ell_B}{\ell_A}, \\ e_2 &= \frac{\ell_A \ell_B (\ell_A^2 - \ell_B^2)}{3(\ell_A^2 + \ell_B^2)^3}, \\ e_4 &= \frac{\ell_A \ell_B (\ell_A^2 - \ell_B^2) (23\ell_A^4 - 354\ell_A^2 \ell_B^2 + 23\ell_B^4)}{60(\ell_A^2 + \ell_B^2)^6}, \\ e_6 &= \frac{\ell_A \ell_B (\ell_A^2 - \ell_B^2) (249\ell_A^8 - 10796\ell_A^6 \ell_B^2 + 40630\ell_B^4 \ell_A^4 - 10796\ell_B^6 \ell_A^2 + 249\ell_B^8)}{126(\ell_A^2 + \ell_B^2)^9}. \end{aligned} \tag{3.36}$$

We see that the shown coefficients e_2, e_4, e_6 vanish when $\ell_A = \ell_B$, and that in fact $e_{2k}(\ell_A, \ell_B) = \frac{1}{2}\pi\delta_{k,0} - e_{2k}(\ell_B, \ell_A), k=0, 1, 2, \dots$. These properties are in agreement with the relations for P in (1.16). Because there is no symmetry in (2.9) with respect to ℓ_A and ℓ_B , they do not follow from the construction of the coefficients d_{2k} and e_{2k} .

When we scale the parameters by putting $\ell_B = \lambda \ell_A$, we see that the shown coefficients obey the relation

$$d_{2k} = \mathcal{O}(\ell_A^{-2k}), \tag{3.37}$$

uniformly with respect to $\lambda \geq 0$. When we write (3.30) with a remainder, that is,

$$Q = d_0 + d_2 + \dots + d_{2k-2} + \int_0^\pi f_{2k}(\theta) e^{\psi(\theta)} d\theta, \tag{3.38}$$

a straightforward analysis shows that similarly

$$f_{2k}(\theta) = \mathcal{O}(\ell_A^{-2k}), \tag{3.39}$$

uniformly with respect to $\lambda \geq 0$ and $\theta \in [0, \pi]$. This shows the nature of the uniform asymptotic expansion of Q , and, after integrating, the nature of the expansion for the probability P .

By expanding the coefficients e_k in (3.36) for large ℓ_A with $\ell_A \gg \ell_B$, we obtain the coefficients of the non-uniform expansion of Section 3.1.

4. Three particles and more

For three random walkers A, B, C the probability integral reads

$$P(t_{\ell_A} \leq t_{\ell_B} \leq t_{\ell_C}) = \int_0^\infty dt_A \bar{F}(\ell_A, t_A) \int_{t_A}^\infty dt_B \bar{F}(\ell_B, t_B) \int_{t_B}^\infty dt_C \bar{F}(\ell_C, t_C), \tag{4.1}$$

with the density as in (1.13). That is,

$$P(t_{\ell_A} \leq t_{\ell_B} \leq t_{\ell_C}) = \ell_A \ell_B \ell_C \int_0^\infty \frac{dt_A}{t_A} e^{-t_A} I_{\ell_A}(t_A) \int_{t_A}^\infty \frac{dt_B}{t_B} e^{-t_B} I_{\ell_B}(t_B) \int_{t_B}^\infty \frac{dt_C}{t_C} e^{-t_C} I_{\ell_C}(t_C). \tag{4.2}$$

It gives the probability that particle A reaches site ℓ_A , before particle B reaches ℓ_B , while B reaches site ℓ_B , before particle C reaches ℓ_C .

First we observe that the probability for three particles arriving at the same site ℓ , that is, $\ell_A = \ell_B = \ell_C = \ell$, equals $\frac{1}{3!}$. This easily follows from (cf. (1.17))

$$\int_{t_A}^\infty dt_B \bar{F}(\ell, t_B) \int_{t_B}^\infty dt_C \bar{F}(\ell, t_C) = \frac{1}{2} \left[\int_{t_A}^\infty \bar{F}(\ell, \tau) d\tau \right]^2. \tag{4.3}$$

Substituting this in (4.2), performing another integration by parts, and using (1.12), gives the value $\frac{1}{3!}$. Using the same method we infer that for n particles the probability for all n particles arriving at the same site ℓ equals $\frac{1}{n!}$.

Repeating the steps used for obtaining (2.7), and replacing all Bessel functions by using (2.3), we easily find for (4.3)

$$P(t_{\ell_A} \leq t_{\ell_B} \leq t_{\ell_C}) = \frac{1}{\pi^3} \int_0^\pi d\theta \frac{\sin \theta \sin \ell_C \theta}{1 - \cos \theta} \int_0^\pi d\sigma \frac{\sin \sigma \sin \ell_B \sigma}{2 - \cos \theta - \cos \sigma} \times \int_0^\pi d\tau \frac{\sin \tau \sin \ell_A \tau}{3 - \cos \theta - \cos \sigma - \cos \tau}. \tag{4.4}$$

Evaluating the τ -integral gives

$$P = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{\sin \theta \sin \ell_C \theta}{1 - \cos \theta} \frac{\sin \sigma \sin \ell_B \sigma}{2 - \cos \theta - \cos \sigma} \left(q + \sqrt{q^2 - 1} \right)^{-\ell_A} d\theta d\sigma, \tag{4.5}$$

where $q = 3 - \cos \theta - \cos \sigma$.

From the above analysis it is clear how a similar integral representation can be obtained for n random walkers A_1, A_2, \dots, A_n . The probability can be written in the form of the n -fold integral

$$P(t_{\ell_{A_1}} \leq t_{\ell_{A_2}} \leq \dots \leq t_{\ell_{A_n}}) = \frac{1}{\pi^n} \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_n \prod_{j=1}^n \frac{\sin \theta_j \sin \ell_{A_j} \theta_j}{\tilde{p}_j}, \tag{4.6}$$

where

$$\tilde{p}_j = \sum_{k=j}^n (1 - \cos \theta_k) = 2 \sum_{k=j}^n \sin^2 \frac{1}{2} \theta_k, \quad j = 1, 2, \dots, n. \tag{4.7}$$

Integrating the θ_1 integral gives

$$P = \frac{1}{\pi^{n-1}} \int_0^\pi d\theta_2 \dots \int_0^\pi d\theta_n \left(p + \sqrt{p^2 - 1} \right)^{-\ell_{A_1}} \prod_{j=2}^n \frac{\sin \theta_j \sin \ell_{A_j} \theta_j}{\tilde{p}_j}, \tag{4.8}$$

where

$$p = 1 + \tilde{p}_2 = n - \sum_{j=2}^n \cos \theta_j. \tag{4.9}$$

4.1. Asymptotic approximations for three particles

For large values of ℓ_A the main contributions to the integral in (4.5) come from the origin $\sigma = 0, \theta = 0$. To see this we observe that

$$q + \sqrt{q^2 - 1} = 3 - \cos \theta - \cos \sigma + \sqrt{(2 - \cos \theta - \cos \sigma)(4 - \cos \theta - \cos \sigma)} = 1 + \sqrt{\theta^2 + \sigma^2} + \mathcal{O}(\theta^2, \theta\sigma, \sigma^2), \tag{4.10}$$

and that

$$\left(q + \sqrt{q^2 - 1} \right)^{-\ell_A} = e^{-\ell_A \ln(q + \sqrt{q^2 - 1})} \sim e^{-\ell_A \sqrt{\theta^2 + \sigma^2}}, \tag{4.11}$$

as $\theta, \sigma \rightarrow 0$. We also have

$$\frac{\theta \sin \theta}{1 - \cos \theta} \sim 2, \quad \frac{\sigma \sin \sigma}{2 - \cos \theta - \cos \sigma} \sim \frac{2\sigma^2}{\theta^2 + \sigma^2} \tag{4.12}$$

as $\theta, \sigma \rightarrow 0$.

This motivates us to consider as a first approximation

$$P \sim \frac{4}{\pi^2} \int_0^\pi d\theta \int_0^\pi d\sigma \frac{\sin(\ell_C \theta)}{\theta} \frac{\sin(\ell_B \sigma)}{\sigma} \frac{\sigma^2}{\theta^2 + \sigma^2} e^{-\ell_A \sqrt{\theta^2 + \sigma^2}}, \tag{4.13}$$

where we have used (4.5), (4.11) and (4.12).

Next we use polar coordinates for θ and σ by writing

$$\theta = r \cos \phi, \quad \sigma = r \sin \phi, \quad 0 \leq r \leq \pi, \quad 0 \leq \phi \leq \frac{1}{2}\pi. \tag{4.14}$$

We extend the finite square in the (θ, σ) -plane to the quarter plane and obtain

$$P \sim \frac{4}{\pi^2} \int_0^{\frac{1}{2}\pi} d\phi \int_0^\infty r dr \frac{\sin(\ell_C r \cos \phi)}{r \cos \phi} \frac{\sin(\ell_B r \sin \phi)}{r \sin \phi} \sin^2 \phi e^{-\ell_A r}. \tag{4.15}$$

The r integral can be found in [3, Eq. (3.947)], that is,

$$\int_0^\infty e^{-ar} \sin(br) \sin(cr) \frac{dr}{r} = \frac{1}{4} \ln \frac{a^2 + (b+c)^2}{a^2 + (b-c)^2}, \tag{4.16}$$

and can be proved by differentiation with respect to a . We obtain

$$P \sim \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \tan \phi \ln \frac{\ell_A^2 + (\ell_C \cos \phi + \ell_B \sin \phi)^2}{\ell_A^2 + (\ell_C \cos \phi - \ell_B \sin \phi)^2} d\phi, \tag{4.17}$$

which can be written as

$$P \sim \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \tan \phi \ln \frac{1 + u \cos 2\phi + v \sin 2\phi}{1 + u \cos 2\phi - v \sin 2\phi} d\phi, \tag{4.18}$$

where

$$u = \frac{\ell_C^2 - \ell_B^2}{2\ell_A^2 + \ell_B^2 + \ell_C^2}, \quad v = \frac{2\ell_C \ell_B}{2\ell_A^2 + \ell_B^2 + \ell_C^2}. \tag{4.19}$$

When v is small we can expand

$$\begin{aligned} \ln \frac{1 + u \cos 2\phi + v \sin 2\phi}{1 + u \cos 2\phi - v \sin 2\phi} &= \ln \frac{1 + \frac{v \sin 2\phi}{1 + u \cos 2\phi}}{1 - \frac{v \sin 2\phi}{1 + u \cos 2\phi}} \\ &= 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{v^{2n+1} \sin^{2n+1} 2\phi}{(1 + u \cos 2\phi)^{2n+1}}, \end{aligned} \tag{4.20}$$

which gives

$$P \sim \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{v^{2n+1}}{2n+1} \int_0^{\frac{1}{2}\pi} \tan \phi \frac{\sin^{2n+1} 2\phi}{(1+u \cos 2\phi)^{2n+1}} d\phi. \tag{4.21}$$

This expansion is useful when ℓ_A is large compared with ℓ_B and ℓ_C .

The integral in (4.21) can be written in terms of a Gauss hypergeometric function, and the sum can be written as an Appell function. This does not give further insight, however. We prefer to give a few further estimates.

For examining the convergence of the series in (4.21), observe that

$$\tan \phi \frac{\sin^{2n+1} 2\phi}{(1+u \cos 2\phi)^{2n+1}} \leq \frac{2}{(1-u)^{2n+1}}, \tag{4.22}$$

with

$$1-u = \frac{2\ell_A^2 + 2\ell_B^2}{2\ell_A^2 + \ell_B^2 + \ell_C^2}, \tag{4.23}$$

which is bounded away from 0, unless ℓ_C is much larger than ℓ_A and ℓ_B .

It follows that expansion (4.21) can be viewed as an asymptotic expansion for small values of v for the right-hand side in (4.18).

Of further interest is that when $u = 0$, that is, $\ell_B = \ell_C$ we can evaluate the right-hand side of (4.21) in terms of elementary functions. In fact we obtain by using

$$\int_0^{\frac{1}{2}\pi} \sin^{2n+2} \phi \cos^{2n} \phi d\phi = \frac{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{1}{2})}{\Gamma(2n + 2)}, \tag{4.24}$$

a Gauss hypergeometric function, that can be written as an elementary function

$$P \sim \frac{1}{\pi} v {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; v^2\right) = \frac{1}{\pi} \arcsin v, \quad v = \frac{\ell_B^2}{\ell_A^2 + \ell_B^2}. \tag{4.25}$$

When $\ell_A = \ell_B = \ell_C$ this becomes

$$P \sim \frac{1}{\pi} \arcsin \frac{1}{2} = \frac{1}{6}, \tag{4.26}$$

which is the exact value.

5. Discussion and concluding remarks

We have discussed in this paper a method of considering different simultaneous independent 1D-random walks. This work has been motivated by an attempt to describe the agglomeration of a number of random walkers on a linear chain which will be fixed when they come to occupy nearest neighbour positions on the chain. In treating this problem it turns out to be possible to effect a transformation of coordinates which makes the evolution equation become separable, such that we obtain a product of “one-particle” equations which can then be mathematically treated as independent random walkers as described

in this paper. However, it turns out that this separation is possible only when the jump probabilities in both directions are equal. This is the reason why we have limited ourselves to equal jump probabilities in this work.

For two particles we have given a complete asymptotic description for the case when ℓ_A and/or ℓ_B are large. For three particles we have also given asymptotic results, but a full description becomes a very complicated matter.

Very recently a paper [2] has appeared which treats a related problem (with discrete time steps) by a different method, involving stochastic matrices.

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